Cambridge Assessment International Education

Cambridge International A Level

MATHEMATICS **9709/31**

Paper 3 Pure Mathematics 3 **May/June 2023**

MARK SCHEME

Maximum Mark: 75

Published

This mark scheme is published as an aid to teachers and candidates, to indicate the requirements of the examination. It shows the basis on which Examiners were instructed to award marks. It does not indicate the details of the discussions that took place at an Examiners' meeting before marking began, which would have considered the acceptability of alternative answers.

Mark schemes should be read in conjunction with the question paper and the Principal Examiner Report for Teachers.

Cambridge International will not enter into discussions about these mark schemes.

Cambridge International is publishing the mark schemes for the May/June 2023 series for most Cambridge IGCSE, Cambridge International A and AS Level and Cambridge Pre-U components, and some Cambridge O Level components.

Generic Marking Principles

These general marking principles must be applied by all examiners when marking candidate answers. They should be applied alongside the specific content of the mark scheme or generic level descriptors for a question. Each question paper and mark scheme will also comply with these marking principles.

GENERIC MARKING PRINCIPLE 1: Marks must be awarded in line with: • the specific content of the mark scheme or the generic level descriptors for the question • the specific skills defined in the mark scheme or in the generic level descriptors for the question • the standard of response required by a candidate as exemplified by the standardisation scripts.
GENERIC MARKING PRINCIPLE 2: Marks awarded are always **whole marks** (not half marks, or other fractions).
GENERIC MARKING PRINCIPLE 3: Marks must be awarded **positively**: • marks are awarded for correct/valid answers, as defined in the mark scheme. However, credit is given for valid answers which go beyond the scope of the syllabus and mark scheme, referring to your Team Leader as appropriate • marks are awarded when candidates clearly demonstrate what they know and can do • marks are not deducted for errors • marks are not deducted for omissions • answers should only be judged on the quality of spelling, punctuation and grammar when these features are specifically assessed by the question as indicated by the mark scheme. The meaning, however, should be unambiguous.
GENERIC MARKING PRINCIPLE 4: Rules must be applied consistently, e.g. in situations where candidates have not followed instructions or in the application of generic level descriptors.
GENERIC MARKING PRINCIPLE 5: Marks should be awarded using the full range of marks defined in the mark scheme for the question (however; the use of the full mark range may be limited according to the quality of the candidate responses seen).
GENERIC MARKING PRINCIPLE 6: Marks awarded are based solely on the requirements as defined in the mark scheme. Marks should not be awarded with grade thresholds or grade descriptors in mind.

Mathematics Specific Marking Principles	
1	Unless a particular method has been specified in the question, full marks may be awarded for any correct method. However, if a calculation is required then no marks will be awarded for a scale drawing.
2	Unless specified in the question, answers may be given as fractions, decimals or in standard form. Ignore superfluous zeros, provided that the degree of accuracy is not affected.
3	Allow alternative conventions for notation if used consistently throughout the paper, e.g. commas being used as decimal points.
4	Unless otherwise indicated, marks once gained cannot subsequently be lost, e.g. wrong working following a correct form of answer is ignored (isw).
5	Where a candidate has misread a number in the question and used that value consistently throughout, provided that number does not alter the difficulty or the method required, award all marks earned and deduct just 1 mark for the misread.
6	Recovery within working is allowed, e.g. a notation error in the working where the following line of working makes the candidate's intent clear.

Mark Scheme Notes

The following notes are intended to aid interpretation of mark schemes in general, but individual mark schemes may include marks awarded for specific reasons outside the scope of these notes.

Types of mark

M Method mark, awarded for a valid method applied to the problem. Method marks are not lost for numerical errors, algebraic slips or errors in units. However, it is not usually sufficient for a candidate just to indicate an intention of using some method or just to quote a formula; the formula or idea must be applied to the specific problem in hand, e.g. by substituting the relevant quantities into the formula. Correct application of a formula without the formula being quoted obviously earns the M mark and in some cases an M mark can be implied from a correct answer.

A Accuracy mark, awarded for a correct answer or intermediate step correctly obtained. Accuracy marks cannot be given unless the associated method mark is earned (or implied).

B Mark for a correct result or statement independent of method marks.

DM or **DB** When a part of a question has two or more 'method' steps, the M marks are generally independent unless the scheme specifically says otherwise; and similarly, when there are several B marks allocated. The notation DM or DB is used to indicate that a particular M or B mark is dependent on an earlier M or B (asterisked) mark in the scheme. When two or more steps are run together by the candidate, the earlier marks are implied and full credit is given.

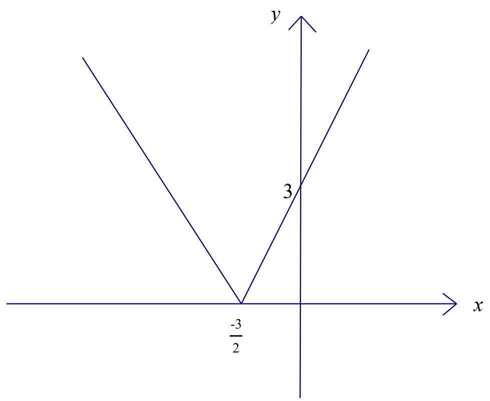
FT Implies that the A or B mark indicated is allowed for work correctly following on from previously incorrect results. Otherwise, A or B marks are given for correct work only.

- A or B marks are given for correct work only (not for results obtained from incorrect working) unless follow through is allowed (see abbreviation FT above).
- For a numerical answer, allow the A or B mark if the answer is correct to 3 significant figures or would be correct to 3 significant figures if rounded (1 decimal place for angles in degrees).
- The total number of marks available for each question is shown at the bottom of the Marks column.
- Wrong or missing units in an answer should not result in loss of marks unless the guidance indicates otherwise.
- Square brackets [ ] around text or numbers show extra information not needed for the mark to be awarded.

Abbreviations

AEF/OE	Any Equivalent Form (of answer is equally acceptable) / Or Equivalent
AG	Answer Given on the question paper (so extra checking is needed to ensure that the detailed working leading to the result is valid)
CAO	Correct Answer Only (emphasising that no 'follow through' from a previous error is allowed)
CWO	Correct Working Only
ISW	Ignore Subsequent Working
SOI	Seen Or Implied
SC	Special Case (detailing the mark to be given for a specific wrong solution, or a case where some standard marking practice is to be varied in the light of a particular circumstance)
WWW	Without Wrong Working
AWRT	Answer Which Rounds To

Question	Answer	Marks	Guidance
1	$3\left(e^{2x}\right)^2 - 5\left(e^{2x}\right) - 4 = 0$	**B1**	OE Form 3 term quadratic in e^{2x}.
	$e^{2x} = \dfrac{5 \pm \sqrt{73}}{6}$, $x = \dfrac{1}{2}\ln\left(\dfrac{5+\sqrt{73}}{6}\right)$	**M1**	Use correct method to solve for x.
	$x = 0.407$	**A1**	Only
		3	

Question	Answer	Marks	Guidance
2(a)		**B1**	Show a recognizable sketch graph of $y = \lvert 2x+3 \rvert$. (Ignore any attempt to sketch $y = 3x + 8$). Straight lines. Vertex in approximately correct position on x axis. Symmetry.
		1	

Question	Answer	Marks	Guidance
2(b)	Find x-coordinate of intersection with $y = 3x + 8$	M1	
	Obtain $x = -\frac{11}{5}$	A1	
	State final answer $x > -\frac{11}{5}$ only	A1	$(x > -2.2)$ Do not condone $\geqslant$ for $>$.
	Alternative Method 1		
	Solve the linear inequality $3x + 8 > -(2x + 3)$, or corresponding linear equation	M1	
	Obtain critical value $x = -\frac{11}{5}$	A1	
	State final answer $x > -\frac{11}{5}$ only	A1	$(x > -2.2)$ Do not condone $\geqslant$ for $>$.
	Alternative Method 2		
	Solve the quadratic inequality $(3x + 8)^2 > (2x + 3)^2$, or corresponding quadratic equation	(M1)	$5x^2 + 36x + 55$.
	Obtain critical value $x = -\frac{11}{5}$	(A1)	Ignore -5 if seen.
	State final answer $x > -\frac{11}{5}$ only	(A1)	$(x > -2.2)$ Do not condone $\geqslant$ for $>$.
		3	

Question	Answer	Marks	Guidance
3	State unsimplified term in x^3, or its coefficient, in the expansion of $(1+4x)^{\frac{1}{2}}$	**B1**	$\dfrac{\frac{1}{2}\times\frac{-1}{2}\times\frac{-3}{2}}{6}(4x)^3\ (=4)$ Must expand binomial coefficient.
	State unsimplified term in x^2, or its coefficient, in the expansion of $(1+4x)^{\frac{1}{2}}$	**B1**	$\dfrac{\frac{1}{2}\times\frac{-1}{2}}{2}(4x)^2\ (=-2)$ Must expand binomial coefficient.
	Multiply by $(3+x)$ and combine terms in x^3, or their coefficients	**M1**	$(3\times 4-1\times 2)$ Allow if they expanded with x rather than $4x$.
	Obtain answer 10	**A1**	Accept $10x^3$
		4	

Question	Answer	Marks	Guidance
4(a)	Use correct double angle formulae	**M1**	e.g. $2\sin\theta\cos\theta+\cos^2\theta-\sin^2\theta=2\sin^2\theta$
	Obtain $\cos^2\theta+2\sin\theta\cos\theta-3\sin^2\theta=0$ from **full and correct** working	**A1**	AG Check conclusion is complete and matches the working.
		2	

Question	Answer	Marks	Guidance
4(b)	Factorise to obtain $(\cos\theta - \sin\theta)(\cos\theta + 3\sin\theta) = 0$	**B1**	OE
	Solve a quadratic in $\sin\theta$ and $\cos\theta$ to obtain a value for θ.	**M1**	$\tan\theta = 1$ *or* $\tan\theta = -\frac{1}{3}$.
	Obtain one correct value e.g. $45°$	**A1**	
	Obtain a second correct value e.g. $161.6°$ and no others in the interval	**A1**	Mark answers in radians (0.785 and 2.82) as a misread. Accept awrt 161.6 .
	Alternative Method 1		
	Obtain $3\tan^2\theta - 2\tan\theta - 1 = 0$	**B1**	
	Solve a 3 term quadratic in $\tan\theta$ to obtain a value for θ.	**M1**	$\tan\theta = 1$ *or* $\tan\theta = -\frac{1}{3}$.
	Obtain one correct value e.g. $45°$	**A1**	
	Obtain a second correct value e.g. $161.6°$ and no others in the interval	**A1**	Mark answers in radians (0.785 and 2.82) as a misread.
	Alternative Method 2		
	Obtain $(\cos\theta + \sin\theta)^2 = (2\sin\theta)^2$	**B1**	
	Solve to obtain a value for θ.	**M1**	$\tan\theta = 1$ *or* $\tan\theta = -\frac{1}{3}$.
	Obtain one correct value e.g. $45°$	**A1**	
	Obtain a second correct value e.g. $161.6°$ and no others in the interval	**A1**	Mark answers in radians (0.785 and 2.82) as a misread.
		4	

Question	Answer	Marks	Guidance
5(a)	State or imply $2xy + x^2\frac{dy}{dx}$ as derivative of $x^2 y$	**B1**	Accept partial: $\frac{\partial}{\partial x} \to 2xy$.
	State or imply $2ay\frac{dy}{dx}$ as derivative of ay^2	**B1**	Accept partial: $\frac{\partial}{\partial y} \to x^2 - 2ay$.
	Equate attempted derivative to zero and solve for $\frac{dy}{dx}$	**M1**	
	Obtain answer $\frac{dy}{dx} = \frac{2xy}{2ay - x^2}$ from correct working	**A1**	AG
		4	
5(b)	State or imply $2ay - x^2 = 0$	***M1**	
	Substitute into equation of curve to obtain equation in x and a or in y and a	**DM1**	e.g. $2ay^2 - ay^2 = 4a^3$ or $\frac{x^4}{2a} - \frac{x^4}{4a} = 4a^3$.
	Obtain one correct point	**A1**	e.g. $(2a, 2a)$.
	Obtain second correct point and no others	**A1**	e.g. $(-2a, 2a)$.
		4	**SC**: Allow A1 A0 for $x = \pm 2a$ or for $y = 2a$.

Question	Answer	Marks	Guidance
6(a)	Obtain a vector for one side of the parallelogram	B1	e.g. $\overrightarrow{AB}=\begin{pmatrix}2\\2\\-1\end{pmatrix}$ or $\overrightarrow{BC}=\begin{pmatrix}-1\\-5\\-6\end{pmatrix}$.
	Correct method to obtain $\pm\overrightarrow{OD}$	M1	e.g. $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{BC}$. MO if use $\overrightarrow{AB}=\overrightarrow{CD}$ or $\overrightarrow{BC}=\overrightarrow{DA}$.
	Obtain $\overrightarrow{OD}=\mathbf{i}-4\mathbf{j}-3\mathbf{k}$	A1	Any equivalent form. Accept coordinates.
		3	
6(b)	Using the correct process, evaluate the scalar product $\overrightarrow{BA}.\overrightarrow{BC}$	M1	$(2+10-6)$ Scalar product of two relevant vectors. OE
	Using the correct process for the moduli, divide the scalar product by the product of the moduli.	M1	$\frac{2+10-6}{\sqrt{9}\times\sqrt{62}}$.
	Obtain answer $\frac{2}{\sqrt{62}}$	A1	ISW Or simplified equivalent i.e. $\frac{\sqrt{62}}{31}$.
		3	

Question	Answer	Marks	Guidance
6(c)	State or imply $\sin\theta = \sqrt{\frac{58}{62}}$	**B1 FT**	Follow *their* $\cos\theta$.
	Use correct method to find the area of *ABCD*	**M1**	**e.g.** $2\times\frac{1}{2}BA\times BC\sin\theta$. Condone decimals.
	Correct unsimplified expression for the area	**A1 FT**	**e.g.** $2\times\frac{1}{2}\times 3\times\sqrt{62}\times\sin\theta$. Condone decimals. Follow *their* sides and angle.
	Obtain answer $3\sqrt{58}$	**A1**	Correct only.
		4	

Question	Answer	Marks	Guidance
7	Correct separation of variables	**B1**	$\int \sin^2 3y\,dy = \int 4\sec 2x \tan 2x\,dx$ or equivalent. Condone missing integral signs or dx and dy.
	Integrate to obtain $k\sec 2x$	**M1**	
	Obtain $2\sec 2x$	**A1**	
	Use double angle formula and integrate to obtain $py + q\sin 6y$	**M1**	Or two cycles of integration by parts.
	Obtain $\frac{1}{2}y - \frac{1}{12}\sin 6y$	**A1**	
	Use $y = 0$, $x = \frac{\pi}{6}$ in a solution containing terms $\lambda\sec 2x$ and $\mu\sin 6y$ to find the constant of integration	**M1**	
	Obtain $\frac{1}{2}y - \frac{1}{12}\sin 6y = 2\sec 2x - 4$	**A1**	Or equivalent seen or implied by $\frac{\pi}{2}\left(-\frac{1}{12}\sin\pi\right) = 2\sec 2x - 4$.
	Obtain $x = 0.541$	**A1**	From correct working (not by using the calculator to integrate).
		8	

Question	Answer	Marks	Guidance
8(a)	State or imply the form $\frac{A}{2x+1}+\frac{B}{x+2}+\frac{C}{(x+2)^2}$	B1	Accept $\frac{A}{2x+1}+\frac{Dx+E}{(x+2)^2}$.
	Use a correct method for finding a constant	M1	
	Obtain one of $A=1, B=-2, C=3$	A1	For alternative form: $A=1, D=-2, E=-1$.
	Obtain a second value	A1	
	Obtain the third value	A1	
		5	
8(b)	Integrate and obtain one of $\frac{1}{2}\ln(2x+1), -2\ln(x+2), \frac{-3}{x+2}$	B1 FT	The follow through is on *their* A, B, C.
	Obtain a second term	B1 FT	If the alternative form is used, then either need to use integration by parts or split the fraction further.
	Obtain the third term	B1 FT	
	Substitute limits correctly in an integral with at least two terms of the form $\frac{1}{2}\ln(2x+1), -2\ln(x+2)$ and $\frac{-3}{x+2}$ and subtract in correct order	M1	The terms used need to have been obtained correctly. Must be exact values, not decimals.
	Obtain $1-\ln 3$	A1	
		5	

Question	Answer	Marks	Guidance
9(a)	Commence integration and reach $pxe^{-2x} + q\int e^{-2x}dx$	*M1	OE
	Obtain $-\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}dx$	A1	OE
	Complete integration and obtain $-\dfrac{1}{2}xe^{-2x} - \dfrac{1}{4}e^{-2x}$	A1	
	Use limits correctly and equate to $\dfrac{1}{8}$, having integrated twice	DM1	$-\dfrac{1}{2}a\,e^{-2a} - \dfrac{1}{4}e^{-2a} + \dfrac{1}{4} = \dfrac{1}{8}$.
	Obtain $a = \dfrac{1}{2}\ln(4a+2)$ correctly	A1	AG
		5	
9(b)	Calculate the values of a relevant expression or pair of expressions at $a = 0.5$ and $a = 1$	M1	
	Justify the given statement with correct calculated values	A1	e.g. $0.5 < 0.69\ldots$, $1 > 0.89\ldots$ $0.193 > 0$, $-1.105 < 0$ $0.066 < 0.125$, $0.148 > 0.125$ if put limits in the integral. Condone if they use calculator for the definite integral.
		2	
9(c)	Use the iterative process $a_{n+1} = \frac{1}{2}\ln(4a_n + 2)$ correctly at least once.	M1	
	Obtain final answer 0.84	A1	
	Show sufficient iterations to at least 4 d.p. to justify 0.84 to 2 d.p. or show that there is a sign change in (0.835, 0.845)	A1	e.g. 0.75, 0.8047, 0.8261, 0.8343, 0.8373, 0.8385 1, 0.8959, 0.8599, 0.8469, 0.8420, 0.8402 .
		3	

Question	Answer	Marks	Guidance
10(a)	Substitute $x = -3$ to obtain value of $\mathrm{p}(-3)$	**M1**	
	Obtain $\mathrm{p}(-3) = 0$ and hence given result	**A1**	
	Alternative method for Question 10(a)		
	Divide $\mathrm{p}(x)$ by $(x+3)$ to obtain quotient $x^2 \pm 2x + \ldots$	**M1**	
	Obtain quotient $x^2 + 2x + 25$, with zero remainder and hence given result	**A1**	
		2	

Question	Answer	Marks	Guidance
10(b)	Substitute $z=-1+2\sqrt{6}\mathrm{i}$ and attempt expansions of z^2 and z^3	**M1**	$z^2=-23-4\sqrt{6}\mathrm{i}$, $z^3=-1+6\sqrt{6}\mathrm{i}+72-48\sqrt{6}\mathrm{i}$.
	Use $\mathrm{i}^2=-1$	**M1**	Seen at least once.
	Obtain $\mathrm{p}(z)=0$ and hence given result	**A1**	**SC B1** if there is no evidence of working for the square or the cube. Total 1/3.
	Alternative Method 1		
	Use roots $z=-1+2\sqrt{6}\mathrm{i}$ to form quadratic factor	**M1**	$z^2+2z+25$.
	Divide $\mathrm{p}(z)$ by *their* quadratic factor	**M1**	
	Obtain zero remainder and hence given result.	**A1**	
	Alternative Method 2		
	Set *their* quadratic factor from **(a)** equal to zero	**M1**	
	Solve for z	**M1**	Need to see method here as answer is given.
	Obtain $z=-1+2\sqrt{6}\mathrm{i}$ (and $z=-1-2\sqrt{6}\mathrm{i}$)	**A1**	
	Alternative Method 3		
	Substitute $z=-1+2\sqrt{6}\mathrm{i}$ into *their* quadratic factor and attempt expansion of z^2	**M1**	
	Use $\mathrm{i}^2=-1$	**M1**	
	Obtain 0 and hence given result	**A1**	
		3	

Question	Answer	Marks	Guidance
10(c)	State $z_1 = \sqrt{3}\mathrm{i}$ and $z_2 = -\sqrt{3}\mathrm{i}$	**B1**	
	Expand $(x+\mathrm{i}y)^2 = -1+2\sqrt{6}\mathrm{i}$ and compare real and imaginary parts	**M1**	Allow for use of $z^2 = -1-2\sqrt{6}\mathrm{i}$.
	Obtain $x^2 - y^2 = -1$ and $xy = \sqrt{6}$	**A1**	
	Solve to obtain x and y	**M1**	
	Obtain $z_3 = \sqrt{2}+\sqrt{3}\mathrm{i}$ and $z_4 = -\sqrt{2}-\sqrt{3}\mathrm{i}$	**A1**	
	Use $z^2 = -1-2\sqrt{6}\mathrm{i}$ to obtain z_5 and z_6	**M1**	Allow for use of $z^2 = -1+2\sqrt{6}\mathrm{i}$.
	Obtain $z_5 = \sqrt{2}-\sqrt{3}\mathrm{i}$ and $z_6 = -\sqrt{2}+\sqrt{3}\mathrm{i}$	**A1**	
		7	